Image Compression Using a Fast and Efficient Discrete Tchebichef Transform Algorithm

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Abstract

The Discrete Tchebichef Transform (DTT) which based on discrete orthogonal Tchebichef polynomials can be an alternative to the Discrete Cosine Transform (DCT) for image processing such as image compression and image recognition as the properties of the DTT are similar to that of the DCT. The DTT not only has higher energy compactness than the DCT in images that have high illumination value variations such as artificial diagrams but also has the advantage of easily computation using a set of recurrence relations.

In this paper, the DTT will be introduced with explanation of its similarity to DCT; also a new fast and efficient 4x4 algorithm for computing the DTT coefficients which can be used in image compression will be proposed. This algorithm reduces computational complexity as measured in terms of the number of arithmetic operations while keeping the accuracy of the reconstructed images.

Keywords: Image Compression, Transform Coding, Discrete Orthogonal Polynomials, the Discrete Tchebichef Transform,

1. Introduction

Images often require a large number of bits to represent them, and if the image needs to be transmitted or stored, it is impractical to do so without somehow reducing the number of bits. Image compression is used to minimize the amount of memory needed to represent an image.

Compressing image in the spatial domain is difficult as typical image's energy often varies significantly throughout the image; however images tend to have a compact representation in the frequency domain packed around the low frequencies which makes compression in the frequency domain more efficient and effective. Transform coding is an image compression technique that first transforms image to the frequency domain, and then compress it. The transform coefficients should be decorrelated, to reduce redundancy and to have a maximum amount of information stored in the smallest space [6].

Moment functions based on discrete orthogonal polynomials are not only used in image compression but also are used as shape descriptors in computer vision applications. Orthogonal moments provide good feature representation capability and improved robustness with respect to image noise, over other types of moments [1: 4].

Tchebichef polynomials are the simplest among discrete orthogonal functions of unit weight, and therefore many of the analytical properties of the corresponding moments can be easily derived, and compared with conventional moments. The Tchebichef polynomials have algebraic recurrence relations involving real coefficients, which make them suitable for defining transform for image compression and reconstruction [1, 2].

The rest of this paper is organized as follow: section (2) is for introducing the Tchebichef Polynomials and section (3) for studying the similarity between the Tchebichef Polynomials and Discrete Cosine Transform. Section (4) for introducing the proposed algorithm and the experimental results of the proposed algorithm are presented in section (5). The last section is preserved to the conclusions.

2. Orthonormal Tchebichef Polynomials

The Discrete Tchebichef Transform (DTT) is a relatively new transform that uses the Tchebichef moments to provide a basis matrix; it is derived from the orthonormal Tchebichef polynomials [2].

Like other transforms, the Discrete Tchebichef Transform (DTT) decorrelates the image data to reduce redundancy between neighboring pixels. After decorrelation each transform coefficient can be encoded independently without losing compression efficiency.

The basis function of the 1-D DTT is defined as the following recurrence relations in polynomials \( t_p(x) \) of degree \( p \) defined on a discrete domain

\[
x = 0, 1\ldots N-1
\]

\[
t_p(x) = (\alpha_1 x + \alpha_2) t_{p-1}(x) + \alpha 3 t_{p-2}(x) \quad p = 2, \ldots, N-1
\]

(1)
Where:
\[
\alpha_1 = \frac{2}{p} \sqrt{4p^2 - 1 - N^2 - p^2}
\]
\[
\alpha_2 = \frac{1-N}{p} \sqrt{4p^2 - 1 - N^2 - p^2}
\]
\[
\alpha_3 = \frac{p-1}{p} \sqrt{\frac{2p+1}{2p-3} \left( N^2 - (p-1)^2 \right) - N^2 - p^2}
\]

The starting values \( t_0(x) \) and \( t_1(x) \) are obtained from the following equations:
\[
t_0(x) = \frac{1}{\sqrt{N}}
\]
\[
t_1(x) = (2x + 1 - N) \frac{3}{N(N^2 - 1)}
\]

The 2-D DTT transform equation can be expressed as:
\[
T_{pq} = \sum_{x=0}^{7} \sum_{y=0}^{7} t_{p(x)} t_{q(y)} f(x,y)
\]
\[
p, q, x, y = 0, 1, \ldots, 7
\]  
(2)

The first transform coefficient is the average value of the sample sequence. In literature, this value is referred to as the DC Coefficient. All other transform coefficients are called the AC Coefficients.

The inverse transformation of DTT which is expressed by equation (2) has the following form:
\[
f(x, y) = \sum_{p=0}^{7} \sum_{q=0}^{7} T_{pq} f_{p(x)} f_{q(y)}
\]
\[
p, q = 0, 1, \ldots, 7
\]  
(3)

Equation (3) can also be expressed using a series representation involving matrices as follows [2]:
\[
f(i, j) = \sum_{p=0}^{7} \sum_{q=0}^{7} T_{pq} G_{pq}(i, j)
\]
\[
p, q = 0, 1, \ldots, 7
\]  
(4)

Where \( G_{pq} \) is an 8x8 matrix (called a basis images) and is defined as:

The 1-D basis images and 2-D basis images for 8x8 discrete Tchebichef polynomials which Used for representing the space via their linear combinations are shown in Figure 1 (a, b) respectively.

\[
G_{pq} =
\]

\[
\begin{bmatrix}
 t_p(0) & t_p(0) & \cdots & t_p(0) \\
 t_p(1) & t_p(1) & \cdots & t_p(1) \\
 \vdots & \vdots & \ddots & \vdots \\
 t_p(7) & t_p(7) & \cdots & t_p(7)
\end{bmatrix}
\]

3. Similarities in the Properties of DTT and DCT

The properties of the Discrete Tchebichef Transform (DTT) which based on discrete orthogonal Tchebichef polynomials are similar to that of the Discrete Cosine Transform (DCT) and it can be a good alternative to DCT

Figure 1: (a) 1-D DTT 8x8 Basis Images; (b) 2-D DTT 8x8 Basis Images
for image processing such as image compression and image recognition. Some of these properties have particular value to image processing applications such as [3-11]:

3.1 Reality

Both of DCT and DTT are real transform, i.e. they have real coefficients.

3.2 Decorrelation

The autocorrelation measures the influence of neighboring pixels on each other. The principle advantage of image transformation is the removal of redundancy between neighboring pixels. This leads to uncorrelated transform coefficients which can be encoded independently (as shown in Figure 2). DTT and DCT exhibit excellent decorrelation properties.

![Figure 2: (a) Normalized autocorrelation of uncorrelated image before and after DCT; (b) Normalized autocorrelation of correlated image before and after DCT.](image1)

3.3 Energy Compaction

Efficiency of a transformation scheme can be gauged by its ability to pack input data into as few coefficients as possible. This allows the quantizer to discard coefficients with relatively small amplitudes without introducing visual distortion in the reconstructed image. DTT and DCT exhibit excellent energy compaction for highly correlated images, the energy of the correlated image is packed into the low frequency region (i.e., top left region). DTT is better than DCT in energy compaction as it is shown in the Figure 3.

![Figure 3: Distribution of DTT and DCT Coefficients of 8x8 Image Block](image2)

The image of DTT and DCT of standard image ('Lena') are shown in Figure 4.

![Figure 4: (a) Lena, (b) Image of its DCT Coefficients and (c) Image of its DTT Coefficients](image3)

3.4 Separability

The DCT transform equation can be expressed as:

\[
C(u, v) = \alpha(u)\alpha(v) \sum_{x=0}^{N-1} \cos \left( \frac{\pi(2x+1)u}{2N} \right) \sum_{y=0}^{N-1} f(x,y) \cos \left( \frac{\pi(2y+1)v}{2N} \right)
\]

And the definition of the DTT can be written in separable form as follows:

\[
T_{pq} = \sum_{x=0}^{N-1} t_p(x) \sum_{y=0}^{N-1} q(y)f(x,y)
\]

Both 2-D DTT and 2-D DCT are just a one-dimensional DTT and DCT respectively applied twice by successive 1-D operations, once in the x direction, and the second in the y direction as shown in Figure 5 and Figure 6 respectively.
3.5 Symmetry

Another look at the row and column operations in Equation (2) reveals that these operations are functionally identical. Such a transformation is called a symmetric transformation. This is an extremely useful property since it implies that the transformation matrix can be precomputed offline and then applied to the image thereby providing orders of magnitude improvement in computation efficiency.

For DCT:

\[ C_m(n) = (-1)^m C_m(N - n - 1) \]

For DTT:

\[ t_p(x) = (-1)^p t_p(N - x - 1) \]

3.6 Orthogonality:

DTT and DCT basis functions are orthogonal thus, the inverse transformation matrix of A is equal to its transpose i.e. \( A^T = A^{-1} \). Therefore, and in addition to its decorrelation characteristics, this property renders some reduction in the pre-computation complexity. Figures 7:10 show the 1-D and 2-D basis images of the DCT and DTT.
4. A Fast DTT Algorithm For 4x4 Image Blocks

Here we propose a fast and efficient algorithm for computing the DTT coefficients for 4x4 image blocks. DTT in matrix form defined as:

\[ T = A \cdot F \cdot A^T \]

Where \( F \) is the \( N \times N \) image matrix and \( A \) is an \( N \times N \) symmetric transformation matrix with entries \( A(i, j) \) given by:

\[ A(p, q) = (\alpha(1 - 2p) + \alpha^2)(1 - 2q) + \alpha^3 \]

The 2-D DTT is just a 1-D DTT applied twice by successive 1-D operations, once in the x-direction, and the second in the y-direction. Matrix-vector product for 1-D DTT may be expressed as:

\[
\begin{bmatrix}
T_0 \\
T_1 \\
T_2 \\
T_3
\end{bmatrix} =
\begin{bmatrix}
H & H & H & H \\
-B & -C & C & B \\
H & -H & -H & H \\
-C & B & -B & C
\end{bmatrix}
\begin{bmatrix}
F_0 \\
F_1 \\
F_2 \\
F_3
\end{bmatrix}
\]

Where:
- \( T_0, ..., T_3 \) are the DTT coefficients (output data)
- \( F_0, ..., F_3 \) are the input data
- \( H = 0.5, B = \frac{\sqrt{5}}{10}, C = \frac{3\sqrt{5}}{10} \)

The top-left function is the basis function of the “DC” coefficient and represents zero spatial frequency. Along the top row the basis functions have increasing horizontal spatial frequency content. Down the left column the functions have increasing vertical spatial frequency content.

For a “typical” image block, most of the DTT coefficients will be near zero. The larger DTT coefficients will be clustered around the DC value (the top-left basis function) i.e. they will be low spatial frequency coefficients. So we developed our algorithm in order to calculate only the most important DTT coefficients, which is the upper left 2x2 corner and consider that the other coefficients will be zeros.

For our proposed algorithm the 2-D DTT is defined to be:

\[ T = Y \cdot F \cdot Y^T \]

Where \( F \) is a \((4 \times 4)\) image block, \( T \) contains the \((2 \times 2)\) 2-D DTT coefficients, and \( Y \) is a proposed \(2 \times 4\) matrix defined as:

\[
Y = \begin{bmatrix}
H & H & H & H \\
-B & -C & C & B
\end{bmatrix}
\]

Define:
- \( K = F_0 + F_1 + F_2 + F_3 \)
- \( M = F_2 - F_0 \)
- \( Z = F_2 - F_1 \)

Removing duplications in the product, the matrix vector product may be expressed as:

\[
\begin{bmatrix}
T_0 \\
T_1
\end{bmatrix} = \begin{bmatrix}
H & 0 & 0 \\
0 & B & C
\end{bmatrix} \cdot \begin{bmatrix}
K \\
M \\
F
\end{bmatrix}
\]

The computation complexity of an image compression algorithm is measured by the number of arithmetic operations to perform both the encoding and decoding process. Table (1) shows the difference between the total number of multiplications and additions needed to transform a 4x4 block by our proposed algorithm and others. Figure 11 illustrates this difference and shows the advantage of our proposed algorithm.

<table>
<thead>
<tr>
<th>Direct Computation</th>
<th>Separability &amp; Symmetry</th>
<th>Ref [10]</th>
<th>Our method</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total No. Of Multiplications</td>
<td>512</td>
<td>128</td>
<td>32</td>
</tr>
<tr>
<td>Total No. Of Additions</td>
<td>240</td>
<td>96</td>
<td>66</td>
</tr>
</tbody>
</table>

Table 1: Total Number of Multiplications and Additions for Different 4x4 2-D DTT Algorithms.
Figure 11: (a) Total Number of Multiplications; (b) Total Number of Additions for Different 4x4 2-D DTT Algorithms.

5. Experimental Results

A set of standard images are used to test the proposed algorithm, the images are first partitioned into non-overlapping blocks of 4x4 pixels (as shown in Figure 12).

It is noticed that the discrete Tchebichef transform gives a better reconstruction for images. In general the discrete Tchebichef transform gives a better performance for images with sharp boundaries, and high predictability.

6. Conclusions

This paper has introduced the Discrete Tchebichef Transform (DTT) which based on discrete orthogonal Tchebichef polynomials. Its properties are similar to that of the Discrete Cosine Transform (DCT) and it can be a good alternative to DCT for image processing such as image compression and image recognition.

The Tchebichef transform involves only algebraic expressions and can be computed easily using a set of recurrence relations. Experimental results have shown that the DTT can provided distinctively better reconstruction compared to DCT as it is higher energy compactness than the DCT in images that have high illumination value variations such as artificial diagrams.

A new fast and efficient algorithm for computing the DTT coefficients for 4x4 image block is proposed. This algorithm reduces the computation complexity as measured in number of multiplications and additions and as a result it can be used efficiently in image compression.
Reference